A PROOF OF GALERKIN'S METHOD FOR EQUATIONS WITH A RETARDING ARGUMENT

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Keldysh [1] has determined the convergence of Galerkin's method for equations without retardation. In the present article his results are applied to the case of a linear equation of the second order with a retarding argument.

Let us examine a homogeneous boundary-value problem of just such an equation:

$$L[y(x)] \equiv \frac{d}{dx}\left[p(x)\frac{dy(x)}{dx}\right] + \lambda\left\{\frac{d}{dx}[q_0(x)\,y(x)] + r_0(x)\,y(x)\right\} + \\ + \sum_{j=1}^{n}\left\{\frac{d}{dx}[q_j(x)\,u(x-h_j(x))] + r_j(x)\,y(x-h_j(x))\right\} = f(x) \tag{1}$$

$$y(x) \equiv 0 \quad \text{on the initial set} \quad E_0, \qquad y(0) = y(1) = 0 \tag{2}$$

Here, $p(x)$, $q_0(x)$, $r_0(x)$, as well as $q_j(x)$, $r_j(x)$ ($j = 1, \ldots, n$) are continuously differentiable functions, given for $0 \leqslant x \leqslant 1$, the function $p(x)$ is zero nowhere; retardations $h_j(x)$ are the continuously non-negative differentiable functions ($0 \leqslant h_j(x) \leqslant 1$); $f(x)$ is a continuous function; λ is the constant complex number from a certain bounded region D of plane λ.

Let us note that requirement $y(x) \equiv 0$ on E_0 is not a restriction of generality, since, if $y(x) = \psi(x) \not\equiv 0$ on E_0, we obtain the present situation through substitution of $y_1(x) = y(x) - \psi(x)$. Under these conditions the right-hand part of (1) will contain a different function $f_1(x)$. We will seek an approximation of the solution of the boundary-value problem (1), (2) in the form of

$$y_m(x) = a_1^{(m)}\varphi_1(x) + \ldots + a_m^{(m)}\varphi_m(x) \tag{3}$$

$$\varphi_k(x) \equiv 0 \quad \text{on} \quad E_0, \qquad \varphi_k(0) = \varphi_k(1) = 0$$

The system of functions

$$1, \varphi_1'(x), \varphi_2'(x), \ldots \tag{4}$$

is complete and normally orthogonal in $L^2_{[0,1]}$ with weight $p(x)$.

We will determine the coefficients $a_k^{(m)}$ $(k = 1, \ldots, m)$ of Expression (3) from the system of equations

$$\int_0^1 L[y_m(x)]\,\varphi_i(x)\,dx = \int_0^1 f(x)\,\varphi_i(x)\,dx \qquad (i = 1, \ldots, m)$$

which may be written in the form of

$$a_i^{(m)} + \sum_{k=1}^{m} (\lambda c_{ik} + d_{ik})\,a_k^{(m)} - f_i = 0 \qquad (i = 1, \ldots, m) \tag{5}$$

where

$$c_{ik} = \int_0^1 [q_0(x)\,\varphi_k(x)\,\varphi_i'(x) - r_0(x)\,\varphi_k(x)\,\varphi_i(x)]dx$$

$$d_{ik} = \sum_{j=1}^{n} \left\{\int_0^1 [q_j(x)\,\varphi_k(x - h_j(x))\;\;\varphi_i'(x) - r_j(x)\,\varphi_k(x - h_j(x))\,\varphi_i(x)]\,dx\right\} \tag{6}$$

$$f_i = -\int_0^1 f(x)\,\varphi_i(x)\,dx$$

We obtain system (5) through truncation of the infinite system of equations

$$a_i + \sum_{k=1}^{\infty} a_k(\lambda c_{ik} + d_{ik}) - f_i = 0 \qquad (i = 1, 2, \ldots) \tag{7}$$

For a proof of the convergence of Galerkin's method it is sufficient to establish that:

1) Solution $y(x)$ of problem (1), (2) and solution $\{a_k\}$ of system (7) are equivalent in the following sense.

a) Solution $a_1, a_2, \ldots$ of system (7) with a convergent sum of squares $a_1^2 + a_2^2 + \ldots < +\infty$ corresponds to each solution $y(x)$ of the problem (1), (2), whereby

$$a_k = \int_0^1 p(x)\,y'(x)\,\varphi_k'(x)\,dx \qquad (k = 1, 2, \ldots) \tag{8}$$

b) To each solution $a_1, a_2, \ldots$ of system (7) with a convergent sum of the squares corresponds solution $y(x)$ of the problem (1), (2), related with $a_1, a_2, \ldots$ by Formulas (8).

2) When $m \to \infty$, the solution of system (7) exists and is a limit of the solutions of the systems of (5) or

$$\lim \sum_{k=1}^{m} \left| a_k - a_k^{(m)} \right|^2 = 0 \quad \text{as} \quad m \to \infty$$

Statement (1) implies that the solution of problem (1), (2) also exists and is unique, wherever the same is true of the system (7) equivalent to it.

Statement (2) implies that the solution of system (7) exists and is unique in all cases where λ is not a characteristic value of a homogeneous system corresponding to system (7).

Accordingly, statements (1) and (2) together imply not only the convergence of Galerkin's method but also the fact itself of the existence and uniqueness of the solution of problem (1), (2).

Proof of statement 1. a) Let $y(x)$ be the solution of problem (1), (2). Then we obtain from Equation (1) and conditions (2)

$$L[y(x)] \equiv f(x) \text{ and } \int_0^1 y'(x)\,dx = y(1) - y(0) = 0 \tag{9}$$

respectively.

Since

$$\int_0^1 \varphi_k'(x)\,dx = 0 \qquad (k = 1, 2, \ldots) \tag{10}$$

and since the system of functions (4) is complete, we may represent $y'(x)$ in the form of a Fourier series

$$y'(x) = \sum_{k=1}^{\infty} a_k \varphi_k'(x) \tag{11}$$

where a_k are found according to Formulas (8). Generally speaking, the series (11) converges in the mean; however, if we integrate it, we obtain a uniformly convergent series:

$$y(x) = \sum_{k=1}^{\infty} a_k \varphi_k(x) \tag{12}$$

If we substitute this series into the identity $L[\,y(x)\,] \equiv f(x)$, we obtain (after having taken into account notations (6) and having integrated by parts) the identities

$$a_i + \sum_{k=1}^{\infty} (c_{ik}\lambda + d_{ik}) a_k - f_i \equiv 0 \qquad (i = 1,2, \ldots) \tag{13}$$

The identities (13) indicate that a_1, a_2, ..., calculated according to Formulas (8), represent the solution of system (7). The convergence of the series $a_1^2 + a_2^2$ ... follows from the fact that the a_k are the coefficients of the converging Fourier series.

b) If a_1, a_2,, ... are now the solution of system (7) and $a_1^2 + a_2^2 + \ldots < \infty$, then, according to the Riesz-Fischer theorem, there exists a certain function $\eta(x)$ which expands into a Fourier series according to the complete system of functions (4) with expansion coefficients a_k

$$\eta(x) = \sum_{k=1}^{\infty} a_k \varphi_k'(x) \tag{14}$$

Let us assume

$$y(x) = \int_0^x \eta(\xi)\, d\xi.$$

Thereupon, $\eta(x) = y'(x)$ and a_k are computed according to Formulas (8). If we now integrate (14), we obtain an expansion $y(x)$ into a uniformly convergent series (12).

Substituting, according to Formulas (8), the expressions for a_i into identities (13), and carrying out the same process for c_{ik}, d_{ik} and f_i according to Formulas (6), we obtain

$$\int_0^1 p(x) y'(x) \varphi_i'(x)\, dx + \sum_{k=1}^{\infty} a_k \left\{\lambda \int_0^1 [q_0(x) \varphi_k(x) \varphi_i'(x) - r_0(x) \varphi_k(x) \varphi_i(x)]\, dx + \right.$$
$$\left. + \sum_{j=1}^{n} \int_0^1 [q_j(x) \varphi_k(x - h_j(x)) \varphi_i'(x) - r_j(x) \varphi_k(x - h_j(x)) \varphi_i(x)]\, dx \right\} + \int_0^1 f(x) \varphi_i(x)\, dx \equiv 0$$

If we alter the order of summation and integration (the series converges uniformly) in the latter identity and take into account (12), we obtain

$$\int_0^1 p(x) y'(x) \varphi_i'(x)\, dx + \lambda \int_0^l [q_0(x) y(x) \varphi_i'(x) - r_0(x) y(x) \varphi_i(x)]\, dx + \tag{15}$$
$$+ \sum_{j=1}^{n} \int_0^1 [q_j(x) y(x - h_j(x)) \varphi_i'(x) - r_j(x) y(x - h_j(x)) \varphi_i(x)]\, dx + \int_0^1 f(x) \varphi_i(x)\, dx \equiv 0$$

$$\int_0^1 \varphi_i'(x)\left\{p(x)y'(x)+\lambda\left[q_0(x)y(x)+\int_0^x r_0(\xi)y(\xi)\,d\xi\right]+\right.$$

$$\left.+\sum_{j=1}^n\left[q_j(x)y(x-h_j(x))+\int_0^x r_j(\xi)y(\xi-h_j(\xi))d\xi\right]-\int_0^x f(\xi)\,d\xi\right\}dx\equiv 0 \qquad (i=1,2,\ldots)$$

Since the system of functions (4) is complete, we obtain from identity (15), taking (10) into account,

$$p(x)y'(x)+\lambda\left[q_0(x)y(x)+\int_0^x r_0(\xi)y(\xi)\,d\xi\right]+$$

$$+\sum_{j=1}^n\left[q_j(x)y(x-h_j(x))+\int_0^x r_j(\xi)y(\xi-h_j(\xi))\,d\xi\right]-\int_0^x f(\xi)\,d\xi=\text{const}$$

Differentiating the latter relation with respect to x, we obtain identity (9), and this indicates that $y(x)$ is a solution of problem (1), (2).

Proof of statement 2. The infinite system of equations (7) has been investigated by Koch in a series of works. Koch's results [2] indicate that the solution of the system of the type (7) exists and is a limit of the sequence of the solutions of system (5), if λ is not a characteristic value of system (7) and of systems (5), and if the series

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}|\lambda c_{ik}+d_{ik}|^2 \tag{16}$$

converges uniformly with respect to λ in region D. Under these conditions, if λ is not a characteristic value of system (7), it is not a value of any of the systems (5).

Consequently, for a proof of statement 2, it is sufficient to establish a uniform convergence of series (16). If $\Lambda=\max|\lambda|$, it is sufficient to demonstrate, for a proof of the uniform convergence of series (16), that

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}|\Lambda c_{ik}+d_{ik}|^2\leqslant|\Lambda|^2\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}|c_{ik}|^2+\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}|d_{ik}|^2<\infty$$

i.e. that the series

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\left|\int_0^1 q_0(x)\varphi_k(x)\varphi_i'(x)\,dx\right|^2, \qquad \sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\left|\int_0^1 r_j(x)\varphi_k(x-h_j(x))\varphi_i(x)dx\right|^2$$

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\left|\int_0^1 r_0(x)\,\varphi_k(x)\,\varphi_i(x)\,dx\right|^2, \qquad \sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\int_0^1 q_j(x)\,\varphi_k(x-h_i(x))\;\;\varphi_i'(x)dx\,\Big|^2$$
$$(j=1,\ldots,n)$$

converge. Let us prove the convergence of the first two series (the convergence of the two other series is proved by analogous arguments). We have

$$\sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\left|\int_0^1 q_0(x)\,\varphi_k(x)\,\varphi_i'(x)\,dx\right|^2 \leqslant \sum_{k=1}^{\infty}\int_0^1 |\,q_0(x)\,\varphi_k(x)\,|^2\frac{dx}{p(x)} \leqslant$$

$$\leqslant \int_0^1 \frac{q_0^2(x)}{p(x)}\left\{\sum_{k=0}^{\infty}\left[\int_0^t \varphi_k'(\xi)\,d\xi\right]^2\right\}dx \leqslant \int_0^1 \frac{q_0^2(x)}{p(x)}\int_0^x \frac{d\xi}{p(\xi)}\,dx = \mathrm{const} < \infty$$

$$\sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\left|\int_0^1 r_j(x)\,\varphi_k(x-h_j(x))\,\varphi_i(x)\,dx\right|^2 \leqslant \sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\int_0^1 |\,r_j(x)\varphi_i(x)|^2 dx$$

$$\cdot\int_0^1 |\,\varphi_k(x-h_j(x))|^2 dx \leqslant \int_0^1 r_j^2(x)\sum_{i=1}^{\infty}\left[\int_0^x \varphi_i(\xi)\,d\xi\right]^2 dx \int_0^1 \sum_{k=1}^{\infty}\left[\int_0^{x-h_j(x)} \varphi_k'(\xi)\,d\xi\right]^2 dx \leqslant$$

$$\leqslant \int_0^1 r_j^2(x)\int_0^x \frac{d\xi}{p(\xi)}\,dx\int_0^1 dx \int_0^{x-h_j(x)} \frac{d\xi}{p(\xi)} = \mathrm{const} < \infty$$

Bessel's and Schwarz's inequalities were employed for the evaluations. Statement 2 is thus proved.

Note. If all retardations are constant ($h_j(x) = h_j = \mathrm{const}$), then, on the basis of the results of [3], the method described above may be used to demonstrate the existence of the solution of a heterogeneous problem for the characteristic values

$$L[y(x)] = 0, \quad y(x) = \psi(x) \quad (-h \leqslant x \leqslant 0,\ h = \max_j h_j), \qquad \psi(0) = y(0) = y(1) = 0$$

and to calculate their approximated values.

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